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A Trade-off between Page Number and Page Width of Book Embeddings of Graphs

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In this paper we describe for any $n \geqslant 3$ a family of graphs possessing a book embedding using n pages—but only with unbounded page width. However, if one uses one additional page, then the page width can be kept bounded by a constant. © 1988 Academic Press, Inc.

A book embedding of a graph is an ordering of its vertices along the spine of a book and an embedding of its edges on the pages so that the edges embedded on the same page do not intersect. The page number of a graph is the minimum number of pages required for a book embedding. The problem of embedding graphs in books was studied first in (Bernhart and Kainen, 1979). Since then, book embedding has been studied by a number of authors. The page number of the planar graphs was examined in (Buss and Shor, 1984). It was shown that any planar graph can be embedded in 9 pages. In (Heath, 1984) this bound was improved to 7 pages. Finally, in (Yannakakis, 1986) it is proved that the page number of the class of planar graphs is precisely 4. In (Heath and Istrail, 1987) it is shown that genus g graphs can be embedded in $O(g)$ pages.

The recent interest in book embeddings of graphs is motivated by an approach to fault-tolerant VLSI design. For more details and further references see (Chung, Leighton, and Rosenberg, 1987; Rosenberg, 1983, 1986).

In devising a book embedding, one strives to minimize both the number of pages used and the page width, i.e., the maximum number of edges crossing any perpendicular to the spine of the book on any page. An algorithm for embedding an n-vertex outerplanar graph in a two-page book with optimal page width $O(\log n)$ was presented in (Heath, 1987). In (Chung, Leighton, and Rosenberg, 1987) it is shown that one-page embeddings of some families of outerplanar graphs require page width $\Omega(n)$. They showed, that in some situations one can achieve a small page number only at the expense of large page width. Indeed, there are families of graphs possessing a book embedding using n pages—but only with unbounded page

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width. However, if one uses one additional page, then the page width can be kept bounded by a constant. In (Chung, Leighton, and Rosenberg, 1987) the authors described such families for $n = 1$ and $n = 2$. They asked whether or not there exist page width–page number tradeoffs for every number of pages n. The aim of this paper is to give a positive answer to this question.

THEOREM. *For any integer $n \geqslant 3$ there is a family of graphs $\{G_{n,k};\ k = 1, 2, ...\}$ such that*

(i) *each $G_{n,k}$ can be embedded in n pages*

(ii) *any family of n-page embeddings of the $G_{n,k}$ has unbounded page widths*

(iii) *there are $(n+1)$-page embeddings of the $G_{n,k}$ such that the page widths are bounded by a constant*

For our purposes it will be convenient to use an alternative description of book embedding. Obviously, a graph can be embedded in n pages if and only if its vertices can be placed on a circle so that its edges are chords of the circle, each chord is assigned one of n colors, and two chords of the same color do not cross. The latter will be called an n-color circle embedding.

Consider the graph $G_{n,k}$ constructed in the following way. Take k copies $K^1_{2n}, ..., K^k_{2n}$ of the complete graph K_{2n} on $2n$ vertices. Denote the vertices of K^i_{2n} by $a_{i,1}, a_{i,2}, ..., a_{i,2n}$ $(i = 1, 2, ..., k)$. Then identify the vertices $a_{i,n}, a_{i+1,1}$ and $a_{i,n+1}, a_{i+1,2n}$ and the corresponding edges $(a_{i,n}, a_{i,n+1})$, $(a_{i+1,1}, a_{i+1,2n})$ $(i = 1, 2, ..., k-1)$. See Fig. 1.

We will show that the graphs $G_{n,k}$ satisfy (i), (ii), and (iii):

(i) It is proved in (Chung, Leighton, and Rosenberg, 1987) that the complete graph K_{2n} on $2n$ vertices is embeddable in n pages and this embedding is optimal in page number. This implies easily that there is an n-color circle embedding of $G_{n,k}$ (Fig. 2).

This proves (i). It is easy to see that the page width of the corresponding

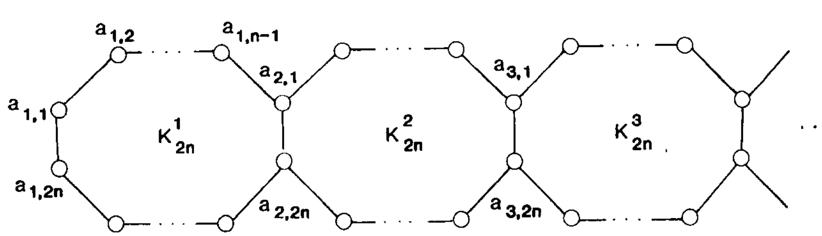

FIGURE 1

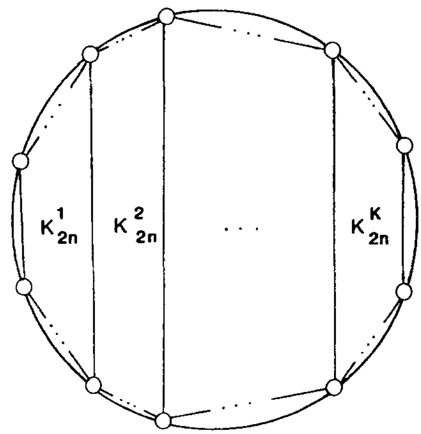

FIGURE 2

book embeddings obtained by cutting the circle between any two vertices and opening it out grows togehter with k, i.e., it is not bounded.

(iii) Now we construct an $(n+1)$-page embedding of $G_{n,k}$ with bounded page width. We decompose the vertices of $G_{n,k}$ into the following blocks:

$$B_0 = (a_{1,1}, a_{1,2n}),$$

$$B_1 = (a_{2,1}, a_{1,n-1}, ..., a_{1,2}, a_{1,2n-1}, ..., a_{1,n+2}, a_{2,2n}), ...,$$

$$B_j = (a_{j+1,1}, a_{j,n-1}, ..., a_{j,2}, a_{j,2n-1}, ..., a_{j,n+2}, a_{j+1,2n}), ...,$$

$$B_k = (a_{k,n}, a_{k,n-1}, ..., a_{k,2}, a_{k,2n-1}, ..., a_{k,n+2}, a_{k,n+1}).$$

This yields a decomposition of the edges of $G_{n,k}$ into intrablock and interblock connections. Intrablock connections are the edges connecting the vertices within one block. Thus, the blocks B_j $(j = 1, ..., k)$ are complete graphs on $2(n-1)$ vertices. For a book embedding of these graphs we need $n-1$ pages.

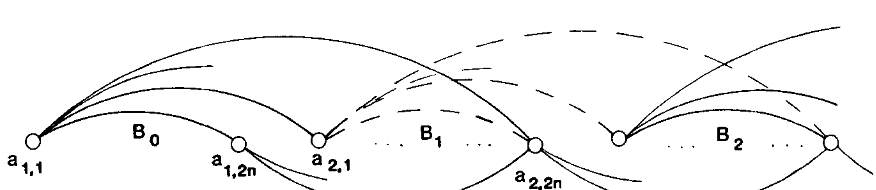

FIGURE 3

Interblock connections are the edges connecting vertices from different blocks. In $G_{n,k}$ we have interblock connections between vertices a_{j1}, $a_{j,2n}$, and all vertices of the block B_j ($j = 1, ..., k$).

Now we place the blocks $B_0, B_1, ..., B_k$ consecutively on a line by preserving the order in which the vertices are written inside the blocks in the definition of $B_0, B_1, ..., B_k$ given above. Then we embed the interblock connections consecutively as follows: the edges connecting vertex $a_{1,1}$ with the vertices of block B_1 are placed on one page, the edges connecting the vertex $a_{1,2n}$ with the vertices of block B_1 are placed on another page, as indicated in Fig. 3. Then we can use the remaining $n-1$ pages to embed the intrablock connections of block B_1.

This procedure can be continued, since the edges connecting vertex $a_{2,1}$ with the vertices of block B_2 can be placed on one of the pages used for the intrablock connections of B_1. Now the edges connecting $a_{2,2n}$ with the vertices of block B_2 can be placed on any other page. Again, we have $n-1$ pages left for the embedding of the intrablock connections of B_2. From Fig. 3 it is easy to see that this embedding has bounded pagewidth.

(ii) Suppose $2n$ points are placed on a circle. A chord is called essential if it connects two points that are not adjacent on the circle.

LEMMA 1. *The maximum number of non-crossing essential chords is* $2n-3$.

Proof. Easy. Induction on n.

LEMMA 2. *Consider an n-color circle embedding of* K_{2n}. *Then*:

(a) *There are exactly* $2n-3$ *essential chords of each color.*

(b) *Any essential chord will necessarily be crossed by chords of all remaining colors.*

(c) *The essential edges (chords) incident to any vertex are labeled by* $n-1$ *colors.*

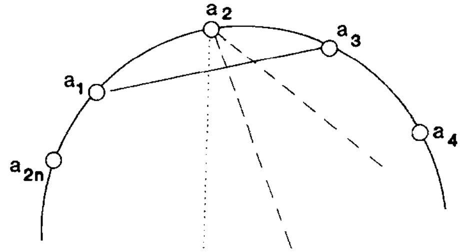

FIGURE 4

Proof. (a) The number of essential chords in any n-color circle embedding of K_{2n} is $n(2n-3)$. Hence, by definition of n-color circle embedding and Lemma 1, we have exactly $2n-3$ essential chords of each color.

(b) It follows from Lemma 1 and (a), that we cannot change the color of any essential chord. This implies (b).

(c) Let the vertices $a_1, a_2, ..., a_{2n}$ of K_{2n} be placed consecutively on a circle, as indicated in Fig. 4. We will show that the essential chords incident to a_2 are labeled by $n-1$ colors. Then, by symmetry, the same holds for all other vertices.

It follows from (b) that the essential chord (a_1, a_3) will necessarily be crossed by chords of $n-1$ remaining colors. But any essential chord crossing (a_1, a_3) is incident to a_2. Thus, the essential chords incident to a_2 are labeled by $n-1$ colors.

Now we proceed to the proof of point (ii) of the theorem. Let an n-color circle embedding of $G_{n,k}$ be given. Consider the complete subgraph on $a_{1,1}, ..., a_{1,2n}$, i.e., K_{2n}^1. These vertices, being placed on the circle, subdivide the circle into arcs. We wish to determine on which of these arcs the vertices $a_{2,2}, ..., a_{2,2n-1}$ may be placed. First we show that these vertices must be placed on at most two neighbouring arcs. Assume for contradiction, that at least two vertices of K_{2n}^1 are placed between $a_{2,i}$, $a_{2,j}$, $i \neq j$, as indicated in Fig. 5, and suppose that the chord $(a_{2,i}, a_{2,j})$ is labeled by color 1.

Then it is easily seen from Fig. 5, that the essential chord $(a_{1,q}, a_{1,s})$ of K_{2n}^1 cannot be labeled by color 1. On the other hand, by Lemma 2(a) in K_{2n}^1 we have exactly $2n-3$ essential chords of color 1 and by Lemma 1 this number is maximal. However, in our case the number of chords of K_{2n}^1 labeled by color 1 cannot be maximal, since $(a_{1,q}, a_{1,s})$, regarded as a chord of an n-color circle embedding of K_{2n}^1, can be relabeled by color 1, a contradiction.

Since $a_{2,2}, ..., a_{2,2n-1}$ are connected one with another, it follows that all these vertices are placed on two neighbouring arcs.

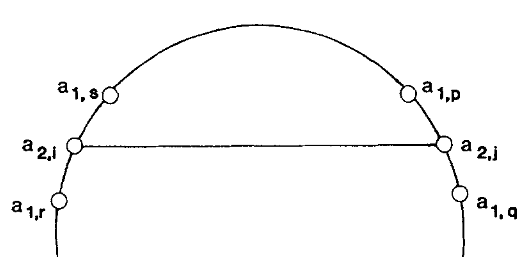

FIGURE 5

Now we show that, in fact, all vertices $a_{2,2}, \ldots, a_{2,2n-1}$ are placed on one arc. On the contrary, suppose that a vertex $a_{1,i}$ is placed between two of these vertices. We have the following cases:

(a) One particular vertex $a_{2,j}$ is placed on one arc and the remaining $2n-3$ vertices are placed on the other arc.

(b) There are at least two vertices on any arc.

Note, that by Lemma 2(c) the adjacent chords of $a_{1,i}$ are labeled by at least $n-1$ colors. Hence, in case (a) all chords connecting $a_{2,j}$ with $a_{2,m}$, $m \neq j$, $m = 2, \ldots, 2n-1$, have the same color.

Let $n \geqslant 4$. We consider the complete graph K_{2n-2} on vertices $a_{2,2}, \ldots, a_{2,2n-1}$. By Lemma 2(c), we have that the edges of this graph incident to $a_{2,j}$ are labeled by $n-2 \geqslant 2$ colors, a contradiction.

Let $n=3$. It is easy to see, up to coloring of the chords connecting adjacent vertices on the circle, that there is only one 3-color circle embedding for the complete graph K_6. See Fig. 6.

We show, that case (a) is impossible. For this purpose we denote the vertices of K_6^1 according to the order in which they are placed on the circle by (a, b, c, d, e, f) and the remaining vertices of the graph $G_{3,2}$ by p, q, r, s. The vertices p, q, r, s and two vertices of the subgraph K_6^1 form the complete subgraph K_6^2. Suppose the vertex p is placed between the vertices a and b, and the vertices q, r, s are placed between b and c. Then the vertex p is connected with q, r, s by chords labeled by color 1. See Fig. 7. The subgraph K_6^2 must have a 3-color circle embedding similar to the embedding in Fig. 6. Consider the essential chords of K_6^2 labeled by color 1. Obviously, the chords (p, r) and (p, s) are essential and the third essential chord labeled

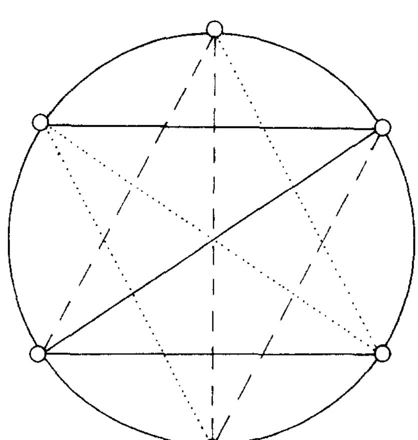

FIGURE 6

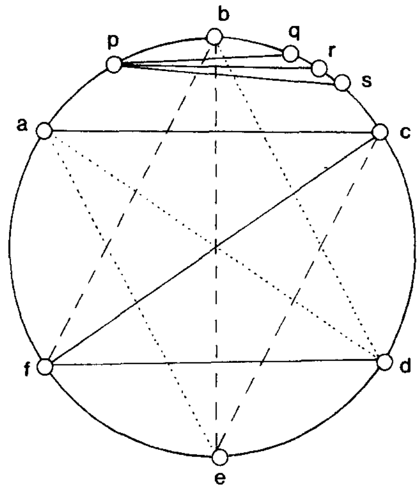

FIGURE 7

by color 1 must be the chord (a, s). Hence, the vertex a is one of the vertices of K_6^2. But now it follows from Fig. 7, that it is impossible to label the chord (a, r) by one of our 3 colors, a contradiction.

In case (b) we take any two points from each arc. Then the complete graph K_4 on these vertices can be labeled by using only one color, a contradiction. Therefore, all vertices $a_{2,2}, ..., a_{2,2n-1}$ are placed on one arc.

Now consider a complete graph on $2n$ vertices $a_{1,n}, a_{2,2}, ..., a_{2,2n-1}, a_{1,n+1}$. Using Lemma 2(c) it is easy to prove that the only possibility for an n-color circle embedding of this graph is to place $a_{2,2}, ..., a_{2,2n-1}$ on the arc between $a_{1,n}$ and $a_{1,n+1}$.

The same arguments, applied successively to $a_{3,2}, ..., a_{3,2n-1}$ and so on, show that one necessarily gets an embedding similar to that of point (i) (Fig. 2). Consequently, the page width grows together with k and this completes the proof of the theorem.

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References

BERNHART, F. AND KAINEN, P. C. (1979), The book thickness of a graph, *J. Combin. Theory Ser. B* **27**, 320–331.

BUSS, J. AND SHOR, P. (1984), On the pagenumber of planar graphs, *in* "16th Annual ACM Symp. on Theory of Computing," pp. 98–100.

CHUNG, F. R. K., LEIGHTON, F. T., AND ROSENBERG, A. L. (1987), Embedding graphs in books: A layout problem with application to VLSI design, *SIAM J. Algebraic Discrete Methods* **8**, 33–58.

HEATH, L. S. (1984), Embedding planar graphs in seven pages, *in* "25th IEEE Symp. on Foundations of Computer Science," pp. 74–83.

HEATH, L. S. (1987), Embedding outerplanar graphs in small books, *SIAM J. Algebraic Discrete Methods* **8**, 198–218.

HEATH, L. S., AND ISTRAIL S. (1987), The pagenumber of genus g graphs is $O(g)$, *in* "19th ACM Symposium on Theory of Computing," pp. 388–397.

ROSENBERG, A. L. (1983), The Diogenes approach to testable fault-tolerant arrays of processors, *IEEE Trans. Comput.* **C-32**, 902–910.

ROSENBERG, A. L. (1986), The Diogenes design methodology: Toward automatic physical layout, *in* "Parallel Algorithms & Architectures," pp. 335–348, North-Holland, Amsterdam.

YANNAKAKIS, M. (1986), Four pages are necessary and sufficient for planar graphs, *in* "18th Annual ACM Symp. on Theory of Computing," pp. 104–108.